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A connection between operator orderings and representations of the Lie algebra $\mathfrak{sl}_2$

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Abstract

We investigate special classes of polynomials in the quantum mechanical position and momentum operators arising from various operator orderings, in particular from the so-called μ-orderings generalizing well-known operator orderings in quantum mechanics such as the Weyl ordering, the normal ordering, etc. Viewing orderings as maps from the polynomial algebra on the phase space to the Weyl algebra generated by the quantum mechanical position and momentum operators we formulate conditions under which these maps intertwine certain naturally defined actions of the Lie algebra $\mathfrak{sl}_2$. These conditions arise via certain regularities in coefficients defining the orderings which can nicely be described in terms of some combinatorial objects called here 'inverted Pascal diagrams'. At the end we establish a connection between radial elements in the Weyl algebra and certain polynomials of the 'number operator' expressible in terms of the hypergeometric function. This is related to another representation of $\mathfrak{sl}_2$, realized in terms of difference operators.

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1. Introduction

In this paper, we investigate possible solutions of the ordering problem under the condition of covariance with respect to the action of $\mathfrak{sl}_2$ Lie algebra derived from the metaplectic representation. For the purpose of this introductory discussion we shall use the phrase 'operator ordering' as a description for some specified procedure, which assigns a well-defined polynomial in the quantum mechanical operators $\hat{p}$ and $\hat{q}$ satisfying the relation $[\hat{q}, \hat{p}] = \mathrm{i}$ to a polynomial in the classical phase-space variables p, q.

The interest in the ordering problem persists since the early days of the quantum mechanics—see Wolf's review [20] of the subject till the mid 1970s of the 20th century.

Our interest in the problem was stimulated, on one hand, by the viewpoint of Howe, who explores the role that dual pairs play in the structure of the Weyl algebra [12, 13], and on the other hand by the discovery of various connections of this problem with the special functions theory, among others done in the papers [2, 3, 5, 15, 16]. Verçin in a recent paper [18] systematically investigated properties of a class of polynomials connected with a certain one-parameter family of orderings, while independently although somewhat later [10] that class of polynomials was investigated from the viewpoint exposed in the present paper by Gnatowska in her Doctor's Thesis at the University of Warsaw—cf also [11].

It has previously been observed by several authors, cf e.g. [5, 3, 18] that some prescriptions for the operator orderings give rise to realizations of representations of the Lie algebra $\mathfrak{sl}_2$ acting on the space of quantum mechanical operators depending polynomially on $\hat{p}$ and $\hat{q}$. Here, we extend these investigations somewhat further by establishing a close relationship between coefficients entering into a general ordering formula, cf (5.1) below, of a product $\hat{p}^n\hat{q}^m$ and the covariance properties with respect to the transferred action of the Lie algebra $\mathfrak{sl}_2$ on the polynomial algebra of classical phase-space variables p and q.

A brief description of results of the paper is in order here. After setting up preliminaries in sections 2 and 3, which include a definition of an important family of μ-orderings, we introduce in section 4 our main device for constructing orderings compatible with an action of the Lie algebra $\mathfrak{sl}_2$. It is a recursively defined two-parameter family of real numbers ϕ_k^n, which we call here an 'inverted Pascal diagram' in view of the similarity of its construction with this famous combinatorial object—cf lemma 1. The main result of the paper is theorem 1, in which we show how $\mathfrak{sl}_2$ actions on the Weyl algebra arise from ordering whose coefficients come from an inverted Pascal diagram. Under this action the Weyl algebra decomposes into a direct sum of invariant subspaces which carry the well-known ladder (or lowest weight) representations of $\mathfrak{sl}_2$, cf (5.7). Finally, it is shown how these representations can be built from a certain family of hypergeometric-type polynomials considered in the papers mentioned above.

2. Notation and preliminaries

We identify the Cartesian space $\mathbb{R}^2$ with points denoted by (p, q) to the classical phase space corresponding to one degree of freedom and let

$$\mathcal{P} = \mathcal{P}(\mathbb{R}^2) = \bigoplus_{k=0}^{\infty} \mathcal{P}^k(\mathbb{R}^2) \tag{2.1}$$

be the polynomial algebra with complex coefficients in phase-space variables, where $\mathcal{P}^k = \mathcal{P}^k(\mathbb{R}^2)$ denotes the subspace of homogeneous polynomials of degree $k \in \mathbb{Z}_+$. (Throughout the paper we adhere to the following convention: $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ will denote the set of nonnegative integers and $\mathbb{N} = \{1, 2, \ldots\}$ that of natural numbers.) Below, we shall describe a somewhat unusual realization of the Lie algebra $\mathfrak{sl}_2$ defined in terms of the following differential operators acting on the polynomial algebra $\mathcal{P}$

$$\begin{aligned} \mathcal{R}P(p,q) &= pqP(p,q),\\ \mathcal{L}P(p,q) &= \frac{\partial^2}{\partial p\partial q}P(p,q),\\ \mathcal{E}P(p,q) &= \left(p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} + 1\right)P(p,q). \end{aligned} \tag{2.2}$$

A routine calculation shows that these operators satisfy the commutation relations

$$[\mathcal{R}, \mathcal{L}] = -\mathcal{E}, \qquad [\mathcal{E}, \mathcal{R}] = 2\mathcal{R}, \qquad [\mathcal{E}, \mathcal{L}] = -2\mathcal{L} \tag{2.3}$$

characterizing the Lie algebra $\mathfrak{sl}_2$ in a Cartan basis, where $\mathcal{E}$ is a generator of the Cartan subalgebra.

It may be checked that this action of $\mathfrak{sl}_2$ on $\mathcal{P}$ leaves invariant each of the following subspaces:

$$\begin{aligned} \mathcal{V}_0 &= \mathrm{span}\{p^k q^k \mid k \in \mathbb{Z}_+\}, \\ \mathcal{V}_n &= \mathrm{span}\{p^k q^{n+k} \mid k \in \mathbb{Z}_+\}, \\ \mathcal{V}_{-n} &= \mathrm{span}\{p^{n+k} q^k \mid k \in \mathbb{Z}_+\}, \end{aligned} \tag{2.4}$$

where $n \in \mathbb{N}$ is arbitrary. Moreover it can be checked that with respect to $\mathcal{E}$, the subspaces $\mathcal{V}_{-n}$ and $\mathcal{V}_n$ are the lowest weight modules for $\mathfrak{sl}_2$ corresponding to the weight $n + 1$, while $\mathcal{V}_0$ corresponds to the lowest weight 1. Finally, there is a direct sum decomposition

$$\mathcal{P} = \mathcal{P}(\mathbb{R}^2) = \mathcal{V}_0 \oplus \bigoplus_{n=1}^{\infty} (\mathcal{V}_{-n} \oplus \mathcal{V}_n), \tag{2.5}$$

which is invariant under this action of $\mathfrak{sl}_2$.

We denote by $\mathcal{K} = \{P \in \mathcal{P} \mid \mathcal{L}P = 0\}$ the space of '$\mathcal{L}$-harmonic' polynomials and by $\mathcal{K}^k = \mathcal{K} \cap \mathcal{P}^k$ its subspace of homogeneous degree k elements. Given $P \in \mathcal{P}$ it is clearly sufficient and necessary for P to belong to $\mathcal{K}$, that P is the sum of polynomials depending separately on p and q, i.e. $P = P_1(p) + P_2(q)$, and thus $\dim \mathcal{K}^k = 2$ for $k > 0$ while $\mathcal{K}^0 = \mathbb{C}$. By a simple algebraic argument one easily arrives at the following decomposition,

$$\mathcal{P}^k = \sum_{j=0}^{[k/2]} (pq)^j \mathcal{K}^{k-2j}, \tag{2.6}$$

where by $[k/2]$ we denote the greatest integer not exceeding $k/2$. In other words, equation (2.6) amounts to the statement that any homogeneous polynomial P of degree k can be expressed in the form

$$P(p,q) = a_0 p^k + b_0 q^k + pq\left(a_1 p^{k-2} + b_1 q^{k-2}\right) + \cdots + \begin{cases} (pq)^l (a_l p + b_l q), & \text{for} \quad k = 2l + 1, \\ (pq)^l, & \text{for} \quad k = 2l, \end{cases}$$

with the uniquely determined coefficients a_j, b_j.

3. μ-orderings

By the Weyl algebra, hereafter denoted by $\mathcal{W}$, we mean the associative algebra with unit over complex numbers $\mathbb{C}$ generated by elements $\hat{p}$ and $\hat{q}$ satisfying the canonical commutation relation

$$[\hat{q}, \hat{p}] = \mathrm{i}. \tag{3.1}$$

As usual, we identify complex numbers $z \in \mathbb{C}$ with the corresponding elements $z \cdot 1$ of the center of the Weyl algebra, where 1 is the unit of $\mathcal{W}$.

As is well known, the commutation relation (3.1) underlies a more subtle algebraic structure of the Weyl algebra than that of the polynomial algebra $\mathcal{P}$. In place of the gradation by means of the degree, embodied in equation (2.1) in the latter case, we have the filtration $\mathcal{W}^0 = \mathbb{C} \subset \mathcal{W}^1 \subset \cdots \subset \mathcal{W}^k \subset \cdots$, with $\mathcal{W} = \bigcup_0^\infty \mathcal{W}^k$ and $\mathcal{W}^k$ denoting the subspace of $\mathcal{W}$ spanned by products of not more than k elements of $\{\hat{q}, \hat{p}\}$. This is the crux of the matter and results in the well-known ambiguity of quantization procedures, and in particular in a multitude of ordering maps.

The elements $\hat{p}$ and $\hat{q}$ are identified as quantum mechanical operators of momentum and position, respectively. Given $w \in \mathcal{W}$, we shall denote by $L_w : \mathcal{W} \mapsto \mathcal{W}$, $R_w : \mathcal{W} \mapsto \mathcal{W}$ linear maps of $\mathcal{W}$ obtained by left or right multiplication with w, respectively. Following the *PhD Thesis* of Gnatowska [10] we considered in our paper [11] one-parameter families $\hat{p}^{(\mu)}$, $\hat{Q}^{(\mu)}$ of linear maps of the Weyl algebra $\mathcal{W}$, with $\mu \in [0, 1]$, defined by

$$\hat{p}^{(\mu)} : \mathcal{W} \mapsto \mathcal{W}, \qquad \hat{p}^{(\mu)} = (1-\mu)L_{\hat{p}} + \mu R_{\hat{p}},$$
$$\hat{Q}^{(\mu)} : \mathcal{W} \mapsto \mathcal{W}, \qquad \hat{Q}^{(\mu)} = \mu L_{\hat{q}} + (1-\mu)R_{\hat{q}}.$$

The crucial observation is that for any fixed μ the maps $\hat{p}^{(\mu)}$ and $\hat{Q}^{(\mu)}$ commute, and therefore the effect of substituting $\hat{p}^{(\mu)}$, $\hat{Q}^{(\mu)}$ into any polynomial $\omega(q, p) \in \mathcal{P}$ is unambiguous. This observation justifies the following definition.

Definition 1. *By the μ-ordering we shall mean the map $\mathcal{O}^{\mu} : \mathcal{P}(\mathbb{R}^2) \mapsto \mathcal{W}$ obtained by setting for any polynomial $\omega = \omega(p, q) \in \mathcal{P}$*

$$\mathcal{O}^{\mu}(\omega) = \omega(\hat{p}^{(\mu)}, \hat{Q}^{(\mu)})1. \tag{3.2}$$

Since the left and right multiplications mutually commute, the binomial formula applies and it is easily seen that this definition produces the same formulae as those stated in the paper [18] of Verçin

$$\mathcal{O}^{\mu}(p^m q^n) = \sum_{k=0}^{m} \binom{m}{k} \mu^k (1-\mu)^{m-k} \hat{p}^{m-k} \hat{q}^n \hat{p}^k, \qquad m, n \in \mathbb{Z}_+. \tag{3.3}$$

In particular, for $\mu = 0, \frac{1}{2}, 1$ this prescription coincides, respectively, with the classical cases of the normal, Weyl and antinormal ordering considered in quantum mechanics, cf e.g. [20].

By way of example we write $\mathcal{O}^{\mu}(\omega)$ for a few polynomials of the lowest degree. For quadratic polynomials the only interesting case is that of pq and from (3.3) it immediately follows that

$$\mathcal{O}^{\mu}(pq) = (1-\mu)\hat{p}\hat{q} + \mu\hat{q}\hat{p}.$$

This operator will play a significant role in the latter part of the paper, so it is reasonable to introduce a special notation for it. We set

$$N = N(\mu) = \mathcal{O}^{\mu}(pq) = (1-\mu)\hat{p}\hat{q} + \mu\hat{q}\hat{p}. \tag{3.4}$$

Note that this formula gives for $\mu = \frac{1}{2}$ the symmetrized product $\frac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p})$ which has played an important role in the previouos works [2, 5, 15, 16].

For the elements of the third degree (3.3) gives (again only nontrivial cases are listed)

$$\mathcal{O}^{\mu}(p^2 q) = (1-\mu)^2 \hat{p}^2 \hat{q} + 2\mu(1-\mu)\hat{p}\hat{q}\hat{p} + \mu^2 \hat{q}\hat{p}^2,$$
$$\mathcal{O}^{\mu}(pq^2) = (1-\mu)\hat{p}\hat{q}^2 + \mu\hat{q}^2\hat{p}.$$

For the classical cases of $\mu = 0, \frac{1}{2}, 1$ we get

Classical ordering of elements of degree 3

	0	$\frac{1}{2}$	1
p^2q	$\hat{p}^2\hat{q}$	$\frac{1}{4}(\hat{p}^2\hat{q} + 2\hat{p}\hat{q}\hat{p} + \hat{q}\hat{p}^2)$	$\hat{q}\hat{p}^2$
pq^2	$\hat{p}\hat{q}^2$	$\frac{1}{2}(\hat{p}\hat{q}^2 + \hat{q}^2\hat{p})$	$\hat{q}^2\hat{p}$

For reasons to become clear later on, the elements of the Weyl algebra given by equation (3.3) with $m = n$ will be called radial. We list below a few radial elements of the lowest order in a general μ-ordering:

$$\begin{aligned}
\mathcal{O}^\mu(p^2q^2) &= (1-\mu)^2\hat{p}^2\hat{q}^2 + 2\mu(1-\mu)\hat{p}\hat{q}^2\hat{p} + \mu^2\hat{q}^2\hat{p}^2,\\
\mathcal{O}^\mu(p^3q^3) &= (1-\mu)^3\hat{p}^3\hat{q}^3 + 3\mu(1-\mu)^2\hat{p}^2\hat{q}^3\hat{p} + 3\mu^2(1-\mu)\hat{p}\hat{q}^3\hat{p}^2 + \mu^3\hat{q}^3\hat{p}^3,\\
\mathcal{O}^\mu(p^4q^4) &= (1-\mu)^4\hat{p}^4\hat{q}^4 + 4\mu(1-\mu)^3\hat{p}^3\hat{q}^4\hat{p} + 6\mu^2(1-\mu)^2\hat{p}^2\hat{q}^4\hat{p}^2\\
&\quad + 4\mu^3(1-\mu)\hat{p}\hat{q}^4\hat{p}^3 + \mu^4\hat{q}^4\hat{p}^4,
\end{aligned}$$

which give in the classical cases

Classical ordering of radial elements of degree $\leqslant 8$			
	0	$\frac{1}{2}$	1
pq	$\hat{p}\hat{q}$	$\frac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p})$	$\hat{q}\hat{p}$
p^2q^2	$\hat{p}^2\hat{q}^2$	$\frac{1}{4}(\hat{p}^2\hat{q}^2 + 2\hat{p}\hat{q}^2\hat{p} + \hat{q}^2\hat{p}^2)$	$\hat{q}^2\hat{p}^2$
p^3q^3	$\hat{p}^3\hat{q}^3$	$\frac{1}{8}(\hat{p}^3\hat{q}^3 + 3\hat{p}^2\hat{q}^3\hat{p} + 3\hat{p}\hat{q}^3\hat{p}^2 + \hat{q}^3\hat{p}^3)$	$\hat{q}^3\hat{p}^3$
p^4q^4	$\hat{p}^4\hat{q}^4$	$\frac{1}{16}(\hat{p}^4\hat{q}^4 + 4\hat{p}^3\hat{q}^4\hat{p} + 6\hat{p}^2\hat{q}^4\hat{p}^2 + 4\hat{p}\hat{q}^4\hat{p}^3 + \hat{q}^4\hat{p}^4)$	$\hat{q}^4\hat{p}^4$

It turns out that the coefficients appearing in the ordering formula (3.3) possess certain combinatorial properties which, as will be shown later, are at the origin of certain actions of the Lie algebra $\mathfrak{sl}_2$ on the Weyl algebra $\mathcal{W}$. These properties are best approached by means of the following scheme which we have called an 'inverted Pascal diagram'.

4. A construction and properties of inverted Pascal diagrams

In this section we discuss the recurrence

$$\phi_k^{n-1} = \phi_k^n + \phi_{k+1}^n$$

for $0 \leqslant k \leqslant n$, which arises in our main theorem, theorem 1, presented in detail in the following section. The diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & \phi_0^1 & & \phi_1^1 & & & \\
 & & \phi_0^2 & & \phi_1^2 & & \phi_2^2 & & \\
 & \phi_0^3 & & \phi_1^3 & & \phi_2^3 & & \phi_3^3 & \\
\phi_0^4 & & \phi_1^4 & & \phi_2^4 & & \phi_3^4 & & \phi_4^4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} \tag{4.1}$$

illustrates that this recurrence may be considered as a Pascal triangle in reverse— an attentive reader notes that while in the Pascal triangle each entry is the sum of two entries immediately above it, here it is the sum of two entries immediately below it. We call this structure an inverted Pascal diagram. We have[3]

Lemma 1. *The solution to the recurrence*

$$\phi_k^{n-1} = \phi_k^n + \phi_{k+1}^n \tag{4.2}$$

[3] The following lemma and its proof were kindly communicated to us by an anonymous referee of the paper.

with initial conditions $\phi_0^n = \alpha_n, n \geqslant 0$ *is*

$$\phi_k^n = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \alpha_{n-i}$$

for $0 \leqslant k \leqslant n$.

Proof. Write

$$\phi_{k+1}^n = \phi_k^{n-1} - \phi_k^n.$$

Introducing the shift operators $E_k^+ f(k) = f(k+1)$, $E_n^- f(n) = f(n-1)$, this becomes

$$E_k^+ \phi = E_n^- \phi - \phi = (E_n^- - 1)\phi.$$

Starting from ϕ_0^n, iterating k times yields

$$\phi_k^n = (E_n^- - 1)^k \alpha_n = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \alpha_{n-i}$$

as stated. □

Of main interest in the present article, the sequence of initial values $\alpha_n = (1-\mu)^n$ yields via the above result

$$\phi_k^n = \mu^k (1-\mu)^{n-k}.$$

The coefficients used in (3.3) for the μ-ordering

$$\beta_k^n = \binom{n}{k} \phi_k^n = \binom{n}{k} \mu^k (1-\mu)^{n-k}$$

are the Bernstein basis polynomials, well known in interpolation theory [17].

5. $\mathfrak{sl}_2$-triples and inverted Pascal diagrams

The μ-orderings considered above in (3.3) may be regarded as a specialization of a general solution of an ordering problem

$$\mathcal{O}(p^m q^n) = \sum_{k=0}^{m} \beta_k^m \hat{p}^{m-k} \hat{q}^n \hat{p}^k, \tag{5.1}$$

where the coefficients β_k^m satisfy for every $m \in \mathbb{Z}_+$ the normalization condition $\sum_{k=0}^m \beta_k^m = 1$, cf e.g. the paper of Bender and Dunne [3]. The elements $\mathcal{O}(p^m q^n)$ form a basis of the Weyl algebra $\mathcal{W}$, and by setting $\mathcal{W}_k = \mathcal{O}(\mathcal{V}_k)$ for $k \in \mathbb{Z}$, we obtain the decomposition

$$\mathcal{W} = \bigoplus_{k \in \mathbb{Z}} (\mathcal{W}_k). \tag{5.2}$$

Given any such ordering it is possible to construct a triple $(\mathbf{R}, \mathbf{L}, \mathbf{E})$ of linear maps of the Weyl algebra $\mathcal{W}$ which satisfy the $\mathfrak{sl}_2$ commutation relations

$$[\mathbf{L}, \mathbf{R}] = \mathbf{E}, \qquad [\mathbf{E}, \mathbf{R}] = 2\mathbf{R}, \qquad [\mathbf{E}, \mathbf{L}] = -2\mathbf{L}, \tag{5.3}$$

and such that the ordering map $\mathcal{O} : \mathcal{P} \to \mathcal{W}$ intertwines the actions of $\mathfrak{sl}_2$ in the respective spaces.

We introduce these maps by declaring that on the basis obtained from monomials ordered by (5.1) they are defined by

$$\mathbf{R}[\mathcal{O}(p^m q^n)] = \mathcal{O}(p^{m+1} q^{n+1}) \tag{5.4}$$
$$\mathbf{L}[\mathcal{O}(p^m q^n)] = mn\mathcal{O}(p^{m-1} q^{n-1}) \tag{5.5}$$
$$\mathbf{E}[\mathcal{O}(p^m q^n)] = (m+n+1)\mathcal{O}(p^m q^n) \tag{5.6}$$

and extending the action to $\mathcal{W}$ by linearity. It is clear that relations (5.3) are indeed satisfied.

Let us observe that each of the subspaces $\mathcal{W}_0 = \mathcal{O}(\mathcal{V}_0)$, $\mathcal{W}_n = \mathcal{O}(\mathcal{V}_n)$, $\mathcal{W}_{-n} = \mathcal{O}(\mathcal{V}_{-n})$, will be invariant under the action of the triple $(\mathbf{R}, \mathbf{L}, \mathbf{E})$ defined by (5.4)–(5.6). It follows that the Weyl algebra $\mathcal{W}$ splits into the sum of $\mathfrak{sl}_2$-modules

$$\mathcal{W} = \mathcal{W}_0 \oplus \bigoplus_{n\in\mathbb{N}} (\mathcal{W}_n \oplus \mathcal{W}_{-n}) \tag{5.7}$$

in full analogy with the decomposition of the polynomial algebra $\mathcal{P}(\mathbb{R}^2)$ in (2.5).

Let $\mathcal{H}$ denote the space of $\mathbf{L}$-harmonic polynomials

$$\mathcal{H} = \mathcal{O}(\mathcal{K}) = \{w \in \mathcal{W} \mid \mathbf{L}w = 0\},$$

and let $\mathcal{H}_k = \mathcal{O}(\mathcal{K}_k) = \mathcal{H} \cap \mathcal{W}^k$. In analogy with (2.6) we have for the given ordering $\mathcal{O}$ the following decomposition

$$\mathcal{W}^k = \sum_{j=0}^{[k/2]} \mathbf{R}^j \mathcal{H}_{k-2j}.$$

It can be shown [10] that in general the space $\mathcal{H}_k$ is independent of the ordering $\mathcal{O}$—for μ-orderings it follows immediately from (5.8), and consequently that this decomposition depends only on the ordering $\mathcal{O}$ through the operator $\mathbf{R}$.

Let us now discuss the resulting representation of $\mathfrak{sl}_2$ in the case of the μ-ordering. To give explicit expressions for $(\mathbf{R}, \mathbf{L}, \mathbf{E})$ we first recall that the adjoint action of the Weyl algebra on itself is defined by

$$\mathrm{ad}(w)v = [w, v], \qquad \text{for} \quad v, w \in \mathcal{W}.$$

For any $w \in \mathcal{W}$ the map $\mathrm{ad}(w)$ is a differentiation of the (associative) algebra $\mathcal{W}$—in particular $\mathrm{ad}(\hat{q})\hat{p}^n = \mathrm{i}n\hat{p}^{n-1}$ and similarly $\mathrm{ad}(\hat{p})\hat{q}^n = -\mathrm{i}n\hat{q}^{n-1}$. Moreover $\mathrm{ad}(\hat{q})$ and $\mathrm{ad}(\hat{p})$ commute with each other,

$$\mathrm{ad}(\hat{q})\,\mathrm{ad}(\hat{p}) = \mathrm{ad}(\hat{p})\,\mathrm{ad}(\hat{q}).$$

By a straightforward computation it can be checked that the operator $\mathbf{L}$ defined by

$$\mathbf{L}(w) = \hat{q}\hat{p}w - \hat{q}w\hat{p} - \hat{p}w\hat{q} + w\hat{p}\hat{q} = \mathrm{ad}(\hat{p}) \circ \mathrm{ad}(\hat{q})(w), \qquad \text{for} \quad w \in \mathcal{W}, \tag{5.8}$$

satisfies (5.5) and is, evidently, independent of μ. With the adjoint operators playing the role of the differentiation, the operator $\mathbf{L}$ seems to be a natural generalization of the 'Laplace' operator $\mathcal{L}$, cf (2.2).

It seems impossible in general to construct a linear map $\mathbf{R}$ of $\mathcal{W}$ into itself which will satisfy condition (5.4) in a way independent of the choice of μ. However one can check that given μ, the operator $\mathbf{R}$ defined as follows,

$$\mathbf{R}w = \hat{P}^{(\mu)}\hat{Q}^{(\mu)}w = \mu(1-\mu)\hat{q}\hat{p}w + \mu^2\hat{q}w\hat{p} + (1-\mu)^2\hat{p}w\hat{q} + \mu(1-\mu)w\hat{p}\hat{q}$$

has the required properties. We can also check that the following formula for $\mathbf{E}$

$$\mathbf{E} = -\mathrm{i}\hat{p}^{(\mu)}\,\mathrm{ad}(\hat{q}) + \mathrm{i}\,\mathrm{ad}(\hat{p})\hat{Q}^{(\mu)},$$

defines an operator satisfying (5.6), cf [10, 11].

The following theorem describes a larger class of orderings $\mathcal{O}$ for which the operator $\mathbf{L}$ defined by (5.8) acts on elements $\mathcal{O}(p^m q^n)$ according to (5.5).

Theorem 1. *Given the Weyl algebra $\mathcal{W}$ generated by a pair $\{\hat{p}, \hat{q}\}$ satisfying (3.1) and given a family of coefficients β_k^m with $m \in \mathbb{Z}_+, 0 \leqslant k \leqslant m$ satisfying the normalization condition $\sum_{k=0}^m \beta_k^m = 1$ for every m, define an ordering $\mathcal{O} : \mathcal{P}(\mathbb{R}^2) \mapsto \mathcal{W}$ by means of the formula*

$$\mathcal{O}(p^m q^n) = \sum_{k=0}^{m} \beta_k^m \hat{p}^{m-k} \hat{q}^n \hat{p}^k. \tag{5.9}$$

For $\mathbf{L} : \mathcal{W} \mapsto \mathcal{W}$ given by

$$\mathbf{L} = \mathrm{ad}(\hat{p}) \circ \mathrm{ad}(\hat{q})$$

the formula

$$\mathbf{L}\mathcal{O}(p^m q^n) = nm\mathcal{O}(p^{m-1} q^{n-1})$$

holds if and only if the coefficients β_k^m for all $m \in \mathbb{Z}_+, 0 \leqslant k \leqslant m$ are of the form $\beta_k^m = \binom{m}{k}\phi_k^m$, with ϕ_k^m satisfying the recurrence relation

$$\phi_k^{m-1} = \phi_k^m + \phi_{k+1}^m,$$

that is, ϕ_k^m forming an inverted Pascal diagram. If this condition is satisfied, we let $\mathbf{R} : \mathcal{W} \to \mathcal{W}$ be defined by

$$\mathbf{R}\mathcal{O}(p^m q^n) = \mathcal{O}(p^{m+1} q^{n+1}), \tag{5.10}$$

and set $\mathbf{E} : \mathcal{W} \to \mathcal{W}$ to be

$$\mathbf{E} = [\mathbf{L}, \mathbf{R}]. \tag{5.11}$$

The maps $\mathbf{R}, \mathbf{L}, \mathbf{E}$ defined above satisfy the $\mathfrak{sl}_2$ commutation relations and leave invariant each of the subspaces $\mathcal{W}_k$ for $k \in \mathbb{Z}$.

Proof. By the property of differentiation

$$\mathrm{ad}(\hat{p}) \circ \mathrm{ad}(\hat{q})(\hat{p}^l \hat{q}^n \hat{p}^m) = n(l\hat{p}^{l-1} \hat{q}^{n-1} \hat{p}^m + m\hat{p}^l \hat{q}^{n-1} \hat{p}^{m-1}).$$

Now a straightforward calculation gives

$$\mathbf{L}\mathcal{O}(p^m q^n) = n \sum_{k=0}^{m-1} \left(\beta_{k+1}^m (k+1) + \beta_k^m (m-k)\right) \hat{p}^{m-1-k} \hat{q}^{n-1} \hat{p}^k, \tag{5.12}$$

and for

$$\beta_k^m = \binom{m}{k} \phi_k^m \tag{5.13}$$

the relation $\phi_k^{m-1} = \phi_k^m + \phi_{k+1}^m$ implies that (5.5) holds. To prove that this relation is also necessary it suffices to consider radial elements only, since the coefficients do not depend on n. Now in this case the elements $\{\hat{p}^{m-k} \hat{q}^m \hat{p}^k\}$ are linearly independent, thus comparing (5.12) with the condition

$$\mathbf{L}\mathcal{O}(p^m q^m) = m^2 \mathcal{O}(p^{m-1} q^{m-1}) = m^2 \sum_{k=0}^{m-1} \beta_k^{m-1} \hat{p}^{m-1-k} \hat{q}^{m-1} \hat{p}^k,$$

we see that the coefficients β_k^m satisfy

$$m\beta_k^{m-1} = (k+1)\beta_{k+1}^m + (m-k)\beta_k^m.$$

From (5.13) it follows in turn that ϕ_k^m satisfy the desired recurrence relation, and since the normalization condition implies that $\phi_0^0 = \beta_0^0 = 1$, the proof is completed. □

In the remaining part of the paper we shall detail, adapting to the present context methods of [10, 11], the structure of the subspace $\mathcal{W}_0$ of radial elements for the case of μ-orderings. It will be shown that it is in fact a commutative subalgebra of $\mathcal{W}$, the algebra of polynomials in the operator $N = \mathcal{O}(pq)$.

The basis $\{\mathcal{B}(n)\}$ of the space $\mathcal{W}_0$ of radial elements consists of weight vectors for the action of $\mathfrak{sl}_2$. It is obtained by successive applications of the operator $\mathbf{R}$ to the unit 1 of the Weyl algebra,

$$\mathcal{B}(n) = \mathbf{R}^n(1)$$

and the action of $\mathfrak{sl}_2$ is given by the familiar formulae

$$\mathbf{R}\mathcal{B}(n) = \mathcal{B}(n+1), \qquad \mathbf{L}\mathcal{B}(n) = n^2\mathcal{B}(n-1), \qquad \mathbf{E}\mathcal{B}(n) = (2n+1)\mathcal{B}(n). \tag{5.14}$$

For the case of μ-ordering a few first elements of this basis are

$$\begin{aligned}
&\mathcal{B}(0) = 1,\\
&\mathcal{B}(1) = N = (1-\mu)\hat{p}\hat{q} + \mu\hat{q}\hat{p},\\
&\mathcal{B}(2) = N^2 + (2\mu-1)\mathrm{i}N - \mu(1-\mu),\\
&\mathcal{B}(3) = N^3 + 3(2\mu-1)\mathrm{i}N^2 + [3\mu(1-\mu)-2]N - 2\mu(1-\mu)(2\mu-1)\mathrm{i},
\end{aligned}$$

and so on.

In fact it is not difficult to show that $\mathcal{B}(n)$ is a polynomial w_n of degree n of the variable N—this was observed earlier in the papers [2, 15, 18] with varied degree of generality as concerns the applicable orderings. An inductive proof of this fact uses an observation that $\mathbf{R}$ is obtained from left and right multiplications by $\hat{q}$ and $\hat{p}$ combined with the so-called pull-through relations

$$\hat{p}w(N) = w(N-\mathrm{i})\hat{p}, \qquad \hat{q}w(N) = w(N+\mathrm{i})\hat{q}, \tag{5.15}$$

which hold for arbitrary polynomial $w(N)$.

To determine the precise form of the polynomials representing basis elements one has to proceed with more care. To begin with, we adopt a convention which will enable us to eliminate the imaginary unit 'i' from the final formulae, by setting $\mathrm{i}M = N$, $w_n(\mathrm{i}t) = \mathrm{i}^n v_n(t)$, which gives

$$\mathcal{B}(n) = w_n(N) = w_n(\mathrm{i}M) = \mathrm{i}^n v_n(M). \tag{5.16}$$

The operators $(\mathbf{L}, \mathbf{R}, \mathbf{E})$ are accordingly changed to $\mathcal{L} = -\mathrm{i}\mathbf{L}$, $\mathcal{R} = -\mathrm{i}\mathbf{R}$ and $\mathcal{E} = -\mathbf{E}$. The following proposition describes now the action of the triple $(\mathcal{L}, \mathcal{R}, \mathcal{E})$ on polynomials in one variable (denoted t for convenience).

Proposition 1. *The difference operators on the algebra of polynomials of one variable t given by*

$$\mathcal{X}v(t) = 2\left(t + \left(\tfrac{1}{2} - \mu\right)\right)v(t) \tag{5.17}$$

$$\mathcal{Y}v(t) = (t-\mu)v(t-1) \tag{5.18}$$

$$\mathcal{Z}v(t) = (t + (1-\mu))v(t+1). \tag{5.19}$$

satisfy the $\mathfrak{sl}_2$ *commutation relations*

$$[\mathcal{X}, \mathcal{Y}] = 2\mathcal{Y}, \qquad [\mathcal{X}, \mathcal{Z}] = -2\mathcal{Z}, \qquad [\mathcal{Z}, \mathcal{Y}] = \mathcal{X}. \tag{5.20}$$

The operators $(\mathcal{L}, \mathcal{R}, \mathcal{E})$ *can be expressed as their linear combinations as follows:*

$$\mathcal{R} = \mu(1-\mu)\mathcal{X} + (1-\mu)^2\mathcal{Y} + \mu^2\mathcal{Z} \tag{5.21}$$

$$\mathcal{L} = \mathcal{X} - \mathcal{Y} - \mathcal{Z} \tag{5.22}$$

$$\mathcal{E} = -(1-2\mu)\mathcal{X} + 2(1-\mu)\mathcal{Y} - 2\mu\mathcal{Z}. \tag{5.23}$$

The proof is an easy, but long calculation based on definitions of the operators and pull-through relations.

Remark 1. Using the classical notation for the forward and backward difference operators, i.e.

$$\Delta v(t) = v(t+1) - v(t) \qquad \nabla v(t) = v(t) - v(t-1),$$

we can describe the action of the triple $(\mathcal{L}, \mathcal{R}, \mathcal{E})$ as follows:

$$\mathcal{R}v(t) = \mu^2(t + (1-\mu))\Delta v(t) - (1-\mu)^2(t-\mu)\nabla v(t) + tv(t)$$

$$\mathcal{L}v(t) = -(t + (1-\mu))\Delta v(t) + (t-\mu)\nabla v(t)$$

$$\mathcal{E}v(t) = -2\mu(t + (1-\mu))\Delta v(t) - 2(1-\mu)(t-\mu)\nabla v(t) - v(t).$$

In this way we obtain the representation of the Lie algebra in difference operators which can be compared to well-known realizations of this structure in differential operators.

Now observe that the combination $\mu(1-\mu)\mathcal{L} + \mathcal{R} + \left(\mu - \frac{1}{2}\right)\mathcal{E}$ is proportional to $\mathcal{X}$ and acts on polynomials in t as the multiplication by $t + \left(\frac{1}{2} - \mu\right)1$. Together with (5.14) and (5.16) this leads to the following result.

Proposition 2. *Let the basis elements* $\mathcal{B}(n)$ *be expressed by polynomials* v_n *of degree* n *in the variable* $M = -\mathrm{i}N$ *according to* $\mathcal{B}(n) = \mathrm{i}^n v_n(M)$. *Then the polynomials* $v_n(M)$ *are determined by the following second-order recurrence formula,*

$$v_{n+1}(M) - [M + n(2\mu - 1)]v_n(M) - \mu(1-\mu)n^2 v_{n-1}(M) = 0, \tag{5.24}$$

together with the initial conditions $v_{-1}(M) = 0$, $v_0(M) = 1$ *and can be written in terms of the hypergeometric function* ${}_2F_1$ *as follows,*

$$v_n(M) = (-1)^n n!(1-\mu)^n {}_2F_1\left(M + (1-\mu), -n, 1; \frac{1}{1-\mu}\right). \tag{5.25}$$

The proof of (5.25) based on the contiguous relation for the hypergeometric function (equations (2.5.16) in [1]) can be found in [10, 11].

For the case of $\mu = \frac{1}{2}$, which corresponds to the Weyl ordering, the right-hand side of (5.25) shows that the polynomials $v_n(\mathrm{i}N)$ coincide up to a normalization with the Meixner–Pollaczek polynomials $P_n^{1/2}(N, \pi/2)$, cf [1, p 348] and [14, (1.7.1)], as pointed out in [15]. For more details on relations between Meixner–Pollaczek and other classes of hypergeometric-type orthogonal polynomials and representations of the Lie algebra $\mathfrak{sl}_2$ the reader is referred to a comprehensive discussion of that topic in [7].

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